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Saturation and an Iterative Construction Process

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Let $\{{}^jK\}_{j=1}^{\infty}$ be a sequence of kernels, let $\{a_j\}_{j=1}^{\infty}$ be a sequence of positive numbers, and let f_0 be a measurable function. Setting $J_0 = f_0$, we study the convergence in $L^p(1 \leqslant p \leqslant 2$ and $p = \infty)$ of the sequence of singular integrals $\{J_n\}_{n=1}^{\infty}$ defined inductively by

$$J_n(x) = (a_n/(2\pi)^{1/2}) \int_{-\infty}^{\infty} J_{n-1}(x-t)\, {}^nK(a_n t)dt, \qquad x \in R.$$

The convergence of $\{J_n\}$ in L^∞ finds an application in Bray–Mandelbrojt's "repeated averaging" construction concerning a non quasi-analytic class of functions.

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We will work with the normalized Lebesgue measure on R, the real line $(dm(x) = dx/(2\pi)^{1/2})$, and will follow Butzer [2] for the definitions of the Fourier transform $\hat{f}$ of a function $f \in L^p$ $(1 \leqslant p \leqslant 2)$ and of the Fourier–Stieltjes transform $\check{g}$ of a function $g \in BV$.

For $f \in L^1$, the inverse Fourier transform of f is defined by:

$$(\mathscr{F}^{-1}f)(t) = \int_{-\infty}^{\infty} f(x)\, e^{ixt}\, dm(x), \qquad t \in R.$$

The proofs of the following propositions can be found in Butzer [2].

PROPOSITION 1. *Let $g_1 \in L^1$, $g_2 \in L^p$ $(1 \leqslant p \leqslant 2)$, and $\phi = g_1 * g_2$, then, $\phi \in L^p$, $\hat{\phi} = \hat{g}_1 \cdot \hat{g}_2$, a.e. (everywhere if $p = 1$) and $\| \phi \|_p \leqslant \| g_1 \|_1 \| g_2 \|_p$.*

PROPOSITION 2. *Let $g_1, g_2 \in BV$ and $\psi(x) = (1/(2\pi)^{1/2}) \int_{-\infty}^{\infty} g_1(x-u)\, dg_2(u)$; then, $\psi \in BV$, $\check{\psi} = \check{g}_1 \cdot \check{g}_2$* and $\| \psi \|_{BV} \leqslant \| g_1 \|_{BV} \| g_2 \|_{BV}$.

If $g_2(x) = \int_{-\infty}^{x} h(t)\, dt$, $h \in L^1$ (i.e., g_2 is absolutely continuous), then $\check{\psi} = \check{g}_1 \cdot \hat{h}$, and $\| \psi \|_{BV} \leqslant \| g_1 \|_{BV} \| h \|_1$.

PROPOSITION 3. *Let $g \in BV, f \in L^p$ $(1 \leqslant p \leqslant 2)$, and*

$$h(x) = (1/(2\pi)^{1/2}) \int_{-\infty}^{\infty} f(x-u)\, dg(u);$$

then, $h \in L^p$, $\hat{h} = \hat{f} \cdot \check{g}$ a.e., (everyhwere if $p = 1$) and $\| h \|_p \leqslant \| f \|_p \| g \|_{BV}$.

DEFINITION 1. A function $K \in NL^1$ is called a kernel; by K_a, $a > 0$, we mean the function defined by:

$$K_a(x) = aK(ax), \qquad x \in R.$$

We have the following theorem concerning the saturation class in L^p $(1 \leqslant p \leqslant 2)$ of the singular integral

$$J_{(\rho,f)}(x) = \int_{-\infty}^{\infty} f(x-u)\, K_\rho(u)\, dm(u), \qquad x \in R$$

THEOREM (Butzer). *Let the kernel K be such that there exist $\psi \in NBV$, and constants $c \neq 0$, $\gamma > 0$ satisfying:*

$$(1 - \hat{K}(v))/c \mid v \mid^\gamma = \check{\psi}(v), \qquad v \neq 0.$$

Then, the saturation class for the singular integral $J_{(\rho,\cdot)}$ (with order $\rho^{-\gamma}$) is:

(i) *In L^1, the class of functions f for which there exists $g \in BV$ such that*

$$c \mid v \mid^\gamma \hat{f}(v) = \check{g}(v). \tag{1}$$

(ii) *In L^p $(1 < p \leqslant 2)$, the class of functions f for which there exists $g \in L^p$ such that*

$$c \mid v \mid^\gamma \hat{f}(v) = \hat{g}(v). \tag{2}$$

Proof. See Butzer [2].

We denote the saturation class above by

$$S(K, \rho^{-\gamma}, p), \qquad \text{for} \quad 1 \leqslant p \leqslant 2.$$

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We now consider $\{{}^nK\}_{n=1}^{\infty}$, a sequence of kernels; $\{a_n\}_{n=1}^{\infty}$, a sequence of positive numbers; and f_0, a measurable function. By $\{f_n\}_{n=1}^{\infty}$, we mean the sequence defined by the iterative process

$$f_{n+1} = f_n * {}^{n+1}K_{a_{n+1}}, \qquad n = 0, 1, 2, \ldots$$

We have the following

THEOREM 1. *Suppose there exist constants $\gamma > 0$, $c \neq 0$ and a sequence of functions $\{{}^j\psi\}_{j=1}^{\infty} \subset NBV$ such that for $j \geqslant 1$*

$$(1 - {}^j\hat{K}(u))/c \mid u \mid^{\gamma} = {}^j\check{\psi}(u), \qquad u \in R, \qquad u \neq 0. \tag{3}$$

Moreover, suppose that $\prod_{j=1}^{\infty} \| {}^jK \|_1$ converges.

If $\sum_{j=1}^{\infty} (1/a_j)^{\gamma}$ converges, then, for $f_0 \in S({}^1K, \rho^{-\gamma}, p)$ $(1 \leqslant p \leqslant 2)$, the sequence $\{f_n\}_{n=1}^{\infty}$ converges in L^p to f_{L^p}, and we also have $\| f_{L^p} - f_0 \|_p \leqslant A \sum_{j=1}^{\infty} (1/a_j)^{\gamma}$, where A is a constant independent of $\{a_j\}$.

Proof. We follow Butzer [1].

(i) $p = 1$. Since $f_0 \in S({}^1K, \rho^{-\gamma}, 1)$, by Butzer's theorem, there exists $g \in BV$ such that

$$c \mid v \mid^{\gamma} \hat{f}_0(v) = \check{g}(v), \qquad v \in R. \tag{4}$$

Consider the sequence $\{g_n\}$ defined by

$$g_0 = g,$$

and

$$g_{n+1} = g_n * {}^{n+1}K_{a_{n+1}}, \qquad n = 0, 1, 2, \ldots .$$

By Proposition 2, we have, for $n \geqslant 1$,

$$\| g_n \|_{BV} \leqslant \| g_{n-1} \|_{BV} \| {}^nK_{a_n} \|_1 \leqslant \cdots \leqslant \| g \|_{BV} \prod_{j=1}^{n} \| {}^jK_{a_j} \|_1$$

Since $\| {}^jK_{a_j} \|_1 = \| {}^jK \|_1$, and $\prod_{j=1}^{\infty} \| {}^jK \|_1 < \infty$, there exists a constant $A > 0$ such that $\| g_n \|_{BV} \leqslant \| g \|_{BV} M_n \leqslant A$, $n \geqslant 1$.

Also, if we define

$${}^n\phi_{a_n}(x) = (1/(2\pi)^{1/2}) \int_{-\infty}^{\infty} g_{n-1}(x - u)\, d({}^n\psi(a_n u)), \qquad x \in R,$$

then,

$${}^n\check{\phi}_{a_n}(v) = \check{g}_{n-1}(v)\, {}^n\check{\psi}(v/a_n), \qquad v \in R.$$

Setting ${}^n\psi_{a_n}(x) = {}^n\psi(a_n x)$,

$$\| {}^n\phi_{a_n} \|_{BV} \leqslant \| g_{n-1} \|_{BV} \| {}^n\psi_{a_n} \|_{BV} \leqslant A \| {}^n\psi \|_{BV} = A. \tag{5}$$

Now, as $f_1 = f_0 * {}^1K_{a_1}$, we have $\hat{f}_1 = \hat{f}_0 \cdot {}^1\hat{K}_{a_1}$. Hence, $a_1^{\gamma}(\hat{f}_0 - \hat{f}_1)(v) = a_1^{\gamma}\hat{f}_0(v)[1 - {}^1\hat{K}(v/a_1)]$, $v \in R$.

But by (3) and (4), and taking $j = 1$,

$$a_1^{\gamma}[\hat{f}_0(v)(1 - {}^1\hat{K}(v/a_1))] = \check{g}(v)\, {}^1\check{\psi}(v/a_1) = {}^1\check{\phi}_{a_1}(v).$$

Hence, by the uniqueness theorem on Fourier–Stieltjes transforms

$${}^1\phi_{a_1}(x) = a_1^{\gamma} \int_{-\infty}^{x} [f_0(u) - f_1(u)]\, du, \qquad x \in R,$$

and by (5), $a_1^{\gamma} \| f_0 - f_1 \|_1 = \| {}^1\phi_{a_1} \|_{BV} \leqslant A$.

Thus, $\| f_0 - f_1 \|_1 \leqslant A a_1^{-\gamma}$.

We also have

$$\begin{aligned} c \mid v \mid^{\gamma} \hat{f}_1(v) &= c \mid v \mid^{\gamma} \hat{f}_0(v) \cdot {}^1\hat{K}(v/a_1) \\ &= \check{g}(v) \cdot {}^1\hat{K}(v/a_1) = (g \check{*} {}^1K_{a_1})(v) = (\check{g}_1)(v). \end{aligned}$$

If we suppose by induction

$$c \mid v \mid^{\gamma} \hat{f}_{n-1}(v) = \check{g}_{n-1}(v), \qquad g_{n-1} \in BV \tag{6}$$

then, by Proposition 2,

$$\begin{aligned} c \mid v \mid^{\gamma} f_n(v) &= c \mid v \mid^{\gamma} \hat{f}_{n-1}(v)\, {}^n\hat{K}(v/a_n) \\ &= \check{g}_{n-1}(v)\, {}^n\hat{K}(v/a_n) \\ &= (g_n \check{*} {}^nK_{a_n})(v) = \check{g}_n(v). \end{aligned}$$

As

$$\begin{aligned} f_n &= f_{n-1} * {}^nK_{a_n} \\ \hat{f}_n &= \hat{f}_{n-1} \cdot {}^n\hat{K}_{a_n}, \end{aligned}$$

and

$$[\hat{f}_{n-1} - \hat{f}_n](v) = \hat{f}_{n-1}(v)[1 - {}^n\hat{K}(v/a_n)], \qquad v \in R \tag{7}$$

By (3) and (6) we have

$$a_n^{\gamma}\hat{f}_{n-1}(v)[1 - {}^n\hat{K}(v/a_n)] = \check{g}_{n-1}(v)\, {}^n\check{\psi}(v/a_n) = {}^n\check{\phi}_{a_n}(v).$$

Again, by (7) and the uniqueness theorem, we have

$${}^n\phi_{a_n}(x) = (a_n^{\gamma}/(2\pi)^{1/2}) \int_{-\infty}^{x} [f_{n-1}(u) - f_n(u)]\, du, \qquad x \in R,$$

and $\| f_{n-1} - f_n \|_1 \leqslant A a_n^{-\gamma}$.

As the series $\sum_{j=1}^{\infty} (1/a_j)^\gamma$ converges, $\{f_n\}$ is a Cauchy sequence in L^1 and converges to a limit f_{L^1} in L^1. Moreover,

$$\| f_{L^1} - f_0 \|_1 \leqslant A \sum_{j=1}^{\infty} (1/a_j)^\gamma.$$

(ii) $1 < p \leqslant 2$. The proof is similar to that in (i), using now the relation $c \mid v \mid^\gamma \hat{f}_0(v) = \hat{g}(v)$, $g \in L^p$ and Propositions 1 and 3. Q.E.D.

Remark. It is obvious that for $\{f_n\}$ to converge, the condition $f_0 \in S({}^1K, \rho^{-\gamma}, p)$ could be replaced by $f_r \in S({}^{r+1}K, \rho^{-\gamma}, p)$ for a certain $r \geqslant 1$.

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THEOREM 2. *Suppose there exist constants $\gamma > 0$ and $A > 0$ such that, for any $j \geqslant 1$*

$$\frac{| 1 - {}^j\hat{K}(u)|}{| u |^\gamma} \leqslant A. \tag{8}$$

Moreover, suppose $\prod_{j=1}^{\infty} \| {}^jK \|_1 < \infty$.

If $\sum_{j=1}^{\infty} (1/a_j)^\gamma < \infty$ and there exists $N \geqslant 1$ such that ${}^N\hat{K} \in L^1$, then, for $f_0 \in L^1$, the sequence $\{f_n\}$ converges in L^∞ (to f_{L^∞}).

Proof. We have

$$| 1 - {}^j\hat{K}(u)| \leqslant A \mid u \mid^\gamma.$$

Hence,

$$| 1 - {}^j\hat{K}(u/a_j)| \leqslant A \mid u \mid^\gamma / a_j^\gamma, \qquad j \geqslant 1.$$

The series $\sum_{j=1}^{\infty} | 1 - {}^j\hat{K}(u/a_j)|$ converges uniformly on compact subsets of R. The infinite product $g = \prod_{j=1}^{\infty} {}^j\hat{K}_{a_j}$ is hence defined.

Moreover, setting: $g_m = \prod_{j=1}^{m} {}^j\hat{K}_{a_j}$, we have

$$| g_m | = | g_{m-1} \, {}^m\hat{K}_{a_m} | \leqslant | g_{m-1} | \, \| {}^mK_{a_m} \|_1 = | g_{m-1} | \, \| {}^mK \|_1 .$$

Hence, for $m > n$

$$| g_m | \leqslant | g_n | \prod_{j=n+1}^{m} \| {}^jK \|_1 .$$

As $\prod_{j=1}^{\infty} \| {}^jK \|_1 < \infty$ and ${}^N\hat{K} \in L^1$, there exists a constant $B > 0$ such that $| g_m | \leqslant B \mid g_N \mid$ for $m > N$ and hence, $g_m \in L^1$ for $m \geqslant N$.

By the Lebesgue convergence theorem, $\| g_n - g \|_1 \to 0$ as $n \to \infty$. Now, for $m \geqslant 1$

$$f_m = f_0 * ({}^1K_{a_1} * \cdots * {}^mK_{a_m}),$$

and so,

$$\hat{f}_m = \hat{f}_0 \cdot \prod_{j=1}^{m} {}^j\hat{K}_{a_j} = \hat{f}_0 \cdot g_m .$$

For $m \geqslant N$, we then have

$$f_m = \mathscr{F}^{-1}[\hat{f}_0 \cdot g_m], \qquad \text{a.e.} \tag{9}$$

But

$$\| \mathscr{F}^{-1}(\hat{f}_0 \cdot g_m) - \mathscr{F}^{-1}(\hat{f}_0 \cdot g) \|_\infty$$
$$\leqslant \| \hat{f}_0(g_m - g) \|_1 \leqslant \| \hat{f}_0 \|_\infty \| g_m - g \|_1 \leqslant \| f_0 \|_1 \| g_m - g \|_1 .$$

Hence, $\{f_m\}$ converges in L^∞ and we have

$$f_{L^\infty} = \mathscr{F}^{-1}(\hat{f}_0 \cdot g), \qquad \text{a.e.}$$

Q.E.D.

Remarks. (1) Relation (9) holds everywhere if f_m is continuous. Also, if we set $g_m{}^* = \hat{f}_0 \cdot g_m$, then the condition ${}^N\hat{K} \in L^1$ could be replaced by $g^*_{n_0} \in L^1$ for a certain n_0.

(2) Conditions (3) and (8) could be equivalently expressed by saying that each of the ratios is, respectively, a (L^1, L^1) and (L^2, L^2) multiplier such that the corresponding two sequences of multiplier operators are uniformly bounded (see Butzer [2]).

COROLLARY. *Under the same conditions as Theorem 2, if there also exists ${}^sK$ continuous such that $\{v^j \, {}^s\hat{K}(v)\} \in L^1$ for any $j \geqslant 1$, then, $f_{L^\infty} \in C^\infty$. Moreover, if $\sum_{j=1}^{\infty} (1/a_j) < \infty$ and f_0 and ${}^jK$, $j = 1, 2,\ldots$, have their supports in a bounded set $D \subset R$, then f_{L^∞} too has compact support.*

Proof. We have $g^* \in L^1$ where $g^* = \hat{f}_0 \cdot g$.

There exists a constant A such that for $j \geqslant 1$

$$| v^j g^*(v) | \leqslant A \| f_0 \|_1 \, | v^j \, {}^s\hat{K}(v) |.$$

Hence, $v^j g^*(v) \in L^1$, and with $D^{(j)}$ denoting the jth derivative,

$$D^{(j)}[\mathscr{F}^{-1}(g^*)] = \mathscr{F}^{-1}[(iv)^j \, g^*(v)].$$

Now, by continuity of ${}^sK$, f_s is continuous (property of convolutions). For $m > s$ (9) holds everywhere.

Hence, we have $f_n \to \mathscr{F}^{-1}g^*$ in L^∞ and $f_{L^\infty} \in C^\infty$.

Let Supp(f) denote the support of the function f in R. By a property of convolutions we have:

$$\begin{aligned} \operatorname{Supp}(f_1) &\subset \operatorname{Supp}(f_0) + \operatorname{Supp}({}^1K_{a_1}) \\ &\subset D + (1/a_1)\, D. \end{aligned}$$

Let $I = [-\xi, \xi]$ be a closed interval containing D. We have:

$$D + (1/a_1)D \subset I + (1/a_1)I.$$

In general, if

$$\operatorname{Supp}(f_{n-1}) \subset I + \left[\sum_{j=1}^{n-1} (1/a_j)\right] I,$$

then,

$$\begin{aligned} \operatorname{Supp}(f_n) &\subset \operatorname{Supp}(f_{n-1}) + \operatorname{Supp}({}^nK_{a_n}) \\ &\subset I + \left[\sum_{j=1}^{n-1} (1/a_j)\right] I + (1/a_n)\, D \\ &\subset I + \left[\sum_{j=1}^{n} (1/a_j)\right] I. \end{aligned}$$

Hence, $\operatorname{Supp}(f_{L^\infty}) \subset (1 + \lambda)I$, where $\lambda = \sum_{j=1}^{\infty} (1/a_j)$. Q.E.D.

Remark. It is obvious that the conditions in the first part of the corollary could be replaced by: "f_m is continuous for a certain m and $v^j g^*(v) \in L^1$." By the smoothing properties of convolutions, we know that f_n is then continuous for $n > m$. This remark is used in the following application.

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DEFINITION. Let $\{N_j\}_{j=0}^{\infty}$ be a sequence of positive reals. By $C\{N_j\}$ we denote the class of all $f \in C^\infty$ such that $\| f^{(j)} \|_\infty \leqslant \alpha_f \beta_f{}^j N_j$, $j = 0, 1, 2, \ldots$ where α_f and β_f are positive constants that depend only on f. The class $C\{N_j\}$ is said to be quasi-analytic if $f \in C\{N_j\}$ and $f^{(n)}(0) = 0$ for $n = 0, 1, 2, \ldots$, imply that $f \equiv 0$. Otherwise, $C\{N_j\}$ is called non quasi-analytic.

The Denjoy–Carleman theorem states that a necessary and sufficient condition for $C\{N_j\}$ to be nonquasi-analytic is that

$$\sum_{n=1}^{\infty} (N_{n-1}/N_n) < \infty.$$

As the non quasi-analyticity of a class is equivalent to the existence in this class of a function with compact support (see Rudin [4]), the following proposition, due to Bray and Mandelbrojt [3], shows the sufficiency of the above condition.

PROPOSITION. *Let* $\{N_n\}_{n=0}^{\infty}$ *be a sequence of positive numbers such that*

$$N_0 = 1 \qquad \text{and} \qquad \sum_{n=1}^{\infty} (N_{n-1}/N_n) < \infty.$$

Let $\eta > 0$ *be sufficiently small. We set* $\lambda_1 = \lambda_2 = \eta$ *and for* $n \geqslant 3$, $\lambda_n = N_{n-3}/N_{n-2}$.

If f_0 *is a bounded measurable function with compact support and* $\{f_n\}$ *the sequence of functions defined inductively by*

$$f_n(x) = (1/2\lambda_n) \int_{-\lambda_n}^{\lambda_n} f_{n-1}(x+t)\, dt, \qquad n = 1, 2,\dots . \tag{10}$$

Then, $\{f_n\}$ *converges uniformly to a function with compact support in* $C\{N_j\}$.

Proof. The direct proof is given in [3]. To use the corollary to Theorem 2, we set: for $j \geqslant 1$

$$a_j = 1/\lambda_j\,,$$

and

$$\begin{aligned} {}^jK &= (\pi/2)^{1/2}, && \text{for} \quad |x| \leqslant 1 \\ &= 0, && \text{for} \quad |x| > 1. \end{aligned}$$

Let D be any bounded set such that

$$D \supset [-1, 1] + \operatorname{Supp}(f_0).$$

We see by (10) that f_1 is continuous. Also,

$$g^*(v) = \hat{f}_0(v) \prod_{i=1}^{\infty} (\sin \lambda_i v/\lambda_i v), \qquad v \in R.$$

Hence, for $j \geqslant 1$

$$\begin{aligned} |v^j g^*(v)| &\leqslant |v^j|\, |\hat{f}_0(v)| \prod_{i=1}^{j+2} |\sin \lambda_i v/\lambda_i v| \\ &\leqslant |\hat{f}_0(v)|\, (\sin \eta v/\eta v)^2\, (\lambda_3\lambda_4 \cdots \lambda_{j+2})^{-1} \\ &= |\hat{f}_0(v)|\, (\sin \eta v/\eta v)^2\, N_j\,. \end{aligned}$$

Hence, $v^j g^*(v) \in L^1$, and by the remark to the corollary, $f_{L^\infty} \in C^\infty$, and has compact support. As

$$D^{(j)} f_{L^\infty} = \mathscr{F}^{-1}[(iv)^j \, g^*(v)],$$
$$\| D^{(j)} f_{L^\infty} \|_\infty \leqslant (\| f_0 \|_1/\eta)(\pi/2)^{1/2} \, N_j \, ,$$

and

$$f_{L^\infty} \in C\{N_j\}. \qquad \text{Q.E.D.}$$

Conclusion

The preceding results could be readily generalized to several variables.

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